Existence and Uniqueness Solution for System of Nonlinear Fractional Integro-Differential Equation

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Abstract

This article investigates a system of nonlinear fractional integro – differential equations with initial conditions. Some new existence and uniqueness results are obtained by applying Picard approximation method.

Keywords: Nonlinear Fractional Integro- Differential Equation, Existence and Uniqueness Solution, Caputo Fractional Derivative, Picard Approximation Method.

1 Introduction

In the past few years, fractional order models are found to be more adequate than integer order models for some real word problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Integro-differential equations arise in many engineering and scientific disciplines often as approximation to partial differential equations, which represent much of the continuum phenomena. The existence and uniqueness problems of fractional nonlinear differential and integro-differential equations as a basic theoretical part of some applications are investigated by many authors (see for examples [3], [6], [5]and [7]).

Most of the practical systems are integro-differential equations in nature and hence the study of integro-differential equations is very important . Exact solutions of integral equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science, it can be used to verify the consistency and estimate errors of various numerical, asymptotic, and approximate methods. the fractional integro-differential equations have been studied by many authors ([1],[2],[4],[9],[10],and [11]].

In this work, we considered the following system of nonlinear fractional integro-differential equations of the form:

$$\left.\begin{aligned} x^{\alpha}(t) &= f(t,x(t),y(t)) + \int_0^b k_1(t,s,x(s),y(s))\,ds \\ x(0) &= c_1 \\ y^{\alpha}(t) &= g(t,x(t),y(t)) + \int_0^b k_2(t,s,x(s),y(s))\,ds \\ y(0) &= c_2 \end{aligned}\right\} \qquad (1)$$

Where $0<\alpha\leq 1,\ 0\leq s\leq t\leq b, c_1, c_2\in R$.

Let the vectors functions $f(t,x(t),y(t))$, $g(t,x(t),y(t))$, $k_1(t,s,x(t),y(t))$ and $k_2(t,s,x(t),y(t))$ are defined and continuous in the domain

$$(t,s,x,y)\in[0,b]\times[0,b]\times D_1^*\times D_2^* \qquad (2)$$

Where D_1^* and D_2^* are closed and bounded domains subsets of Euclidean space R^n . Suppose that the vectors functions satisfying the following inequalities:

A_1 There exists constants M_1, M_2 such that $\|f(t,x,y)\|\leq M_1$ *and* $\|k_1(t,s,x,y)\|\leq M_2$

A_2 There exists constants M_3, M_4 such that $\|g(t,x,y)\|\leq M_3$ *and* $\|k_2(t,s,x,y)\|\leq M_4$

A_3 There exists constants L_1, L_2 such that $\| f(t, x_2, y_2) - f(t, x_1, y_1) \| \leq L_1 (\|x_2 - x_1\| + \|y_2 - y_1\|)$

$$and \ \| k_1(t, s, x_2, y_2) - k_1(t, s, x_1, y_1) \| \leq L_2 (\|x_2 - x_1\| + \|y_2 - y_1\|)$$

A_4 There exists constants L_3, L_4 such that $\| g(t, x_2, y_2) - g(t, x_1, y_1) \| \leq L_3 (\|x_2 - x_1\| + \|y_2 - y_1\|)$

$$and \ \| k_2(t, s, x_2, y_2) - k_2(t, s, x_1, y_1) \| \leq L_4 (\|x_2 - x_1\| + \|y_2 - y_1\|)$$

for all $t \in [0, b]$, $x, x_1, x_2 \in D_1^*$ and $y, y_1, y_2 \in D_1^*$, where M_1, M_2, M_3, M_4 are positive constant vectors $, L_1, L_2, L_3$ and L_4 are positive constant matrices , and also $\|.\| = \max_{t \in [0,b]} \|.\|$.

We define the non- empty sets:

$$\left.\begin{aligned} D_{1\alpha} &= D_1 - \frac{(M_1 + M_2 b) b^{\alpha}}{\Gamma(\alpha+1)} \\ D_{2\alpha} &= D_2 - \frac{(M_3 + M_4 b) b^{\alpha}}{\Gamma(\alpha+1)} \end{aligned}\right\} \quad (3)$$

Furthermore, we suppose that the greatest eigenvalue $\lambda_{\alpha \max}$ of the matrix:

$$\Lambda_{\alpha}(t) = \begin{pmatrix} \frac{(L_1 + bL_2) t^{\alpha}}{\Gamma(\alpha+1)} & \frac{(L_1 + bL_2) t^{\alpha}}{\Gamma(\alpha+1)} \\ \frac{(L_3 + bL_4) t^{\alpha}}{\Gamma(\alpha+1)} & \frac{(L_3 + bL_4) t^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix}$$

, does not exceed unity, i.e.:

$$\lambda_{\alpha \max}(\Lambda_{\alpha}(t)) = \left[\frac{(L_1 + bL_2) b^{\alpha}}{\Gamma(\alpha+1)} + \frac{(L_3 + bL_4) b^{\alpha}}{\Gamma(\alpha+1)} \right] < 1 \quad (4)$$

2 Preliminaries

In this section, we introduce some definitions, and some fundamental facts of Caputo's derivative of fractional order (see [13], [14]).

Definition 2.1. Caputo's derivative for a function $f\ :\ 0,\infty\ \to R$ can been written as

$$ {}_{o}^{t}D^{\alpha}f\ \ t\ \ = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{f^{(n)}(t)dt}{t-s\ \ ^{\alpha+1-n}}, \qquad n=\ \alpha\ +1 $$

Where α denotes the integer part of real number α .

Definition 2.2. The integral

$$ {}_{o}^{t}I^{\alpha}f\ \ t\ \ = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{f(t)}{t-s\ \ ^{1-s}}dt\ , \qquad t>0 $$

Where $\alpha>0,$ is called Riemann-Liouville fractional integral of order α .

Definition 2.3. A sequence $f\ \ _{n=1}^{\infty}$ of functions defined on a set D is said to converge uniformly on D if given $\varepsilon>0,$ there is a positive integer number N_0 such that for all $t\in D$ and all $n\geq N_0,$ we have

$$ \left\| f_n(t)-f(t)\right\| <\varepsilon . $$

Lemma 2.1. Let $\alpha>0,$, then:

$$ {}_{a}^{t}D^{-\alpha}\,{}_{a}^{s}D^{\alpha}y(t)=y(t)+c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1}, \qquad \text{for some } c_i\in R\ ,\ i=0,1,2,\ldots,n-1,\ n=\ \alpha\ +1. $$

3 The Main Results.

The study of the existence and uniqueness solution for the system of nonlinear fractional integro-differential equations (1) will be introduced by the theorems:

Theorem 3.1. (Existence theorem)

Let $f(t,x,y), g(t,x,y), k_1(t,s,x,y)$ *and* $k_2(t,s,x,y)$ be vector functions which are defined and continues on the domains (2). If the hypotheses A_1 - A_4 are satisfied, then the system of nonlinear fractional integro-differential equations (1) has a solution $\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$ which defined as:

$$\begin{pmatrix} x(t,x_0,y_0) = c_1 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\Big[f(s,x(s,x_0,y_0),y(s,x_0,y_0)) + \int_0^b k_1(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\,d\tau\Big]ds \\ y(t,x_0,y_0) = c_2 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\Big[f(s,x(s,x_0,y_0),y(s,x_0,y_0)) + \int_0^b k_2(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\,d\tau\Big]ds \end{pmatrix}$$

Proof: Consider the sequences of the functions $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ are continuous and defined by:

$$x_{m+1}(t,x_0,y_0) = c_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))\,ds + \\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b k_1(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0))\,d\tau\,ds \qquad (5)$$

$$y_{m+1}(t,x_0,y_0) = c_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))\,ds + \\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b k_2(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0))\,d\tau\,ds \qquad (6)$$

The proof will be given in several steps.

Step 1: We shall to prove $x_m(t,x_0,y_0) \in D_1^*$ and $y_m(t,x_0,y_0) \in D_2^*$ for all $t \in [0,b]$, $x_0 \in D_{1\alpha}$ and $y_0 \in D_{2\alpha}$

When m= 1 and using A_1 , A_1 , we have:

$$\|x_1(t,x_0,y_0)-c_1\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_0(s,x_0,y_0),y_0(s,x_0,y_0))\|ds + $$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b \|k_1(s,\tau,x_0(\tau,x_0,y_0),y_0(\tau,x_0,y_0))\|d\tau\, ds$$

$$\leq \frac{(M_1+bM_2)t^{\alpha}}{\Gamma(\alpha+1)}$$

that is , $\|x_1(t,x_0,y_0)-c_1\| \leq \frac{(M_1+bM_2)b^{\alpha}}{\Gamma(\alpha+1)}$

therefore

$$\|y_1(t,x_0,y_0)-c_2\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s,x_0(s,x_0,y_0),y_0(s,x_0,y_0))\|ds + $$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b \|k_2(s,\tau,x_0(\tau,x_0,y_0),y_0(\tau,x_0,y_0))\|d\tau\, ds$$

$$\leq \frac{(M_3+bM_4)t^{\alpha}}{\Gamma(\alpha+1)}$$

that is, $\|y_1(t,x_0,y_0)-c_2\| \leq \frac{(M_3+bM_4)b^{\alpha}}{\Gamma(\alpha+1)}$

so that, $x_1(t,x_0,y_0)\in D_1^*$ *and* $y_1(t,x_0,y_0)\in D_2^*$ for all $x_0 \in D_{1\alpha}$, $y_0 \in D_{2\alpha}$, $t \in [0,b]$.

Therefore by mathematical induction, we get

$$\|x_m(t,x_0,y_0)-c_1\| \leq \frac{(M_1+bM_2)b^{\alpha}}{\Gamma(\alpha+1)} \quad \text{and} \quad \|y_m(t,x_0,y_0)-c_1\| \leq \frac{(M_3+bM_4)b^{\alpha}}{\Gamma(\alpha+1)}$$

In the others words

$$x_m(t,x_0,y_0)\in D_1^* \ , \ \forall \ x_0\in D_{1\alpha} \ , t\in [0,b]$$
$$y_m(t,x_0,y_0)\in D_1^* \ , \ \forall \ y_0\in D_{2\alpha} \ , t\in [0,b] \ , m=0,1,2,\ldots$$

Step 2: we claim that the sequences of functions (5) and (6) are uniformly convergent on the domain (2).

For m=1 and using A_3 , A_4 . We find that:

$$\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|\le\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_1(s,x_0,y_0),y_1(s,x_0,y_0))-f(s,x_0(s,x_0,y_0),y_0(s,x_0,y_0))\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b \|k_1(s,\tau,x_1(\tau,x_0,y_0),y_1(\tau,x_0,y_0))-k_1(s,\tau,x_0(\tau,x_0,y_0),y_0(\tau,x_0,y_0))\|d\tau\, ds$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_1\left(\|x_1(s,x_0,y_0)-x_0(s,x_0,y_0)\|+\|y_1(s,x_0,y_0)-y_0(s,x_0,y_0)\|\right)ds+$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b L_2\left(\|x_1(\tau,x_0,y_0)-x_0(\tau,x_0,y_0)\|+\|y_1(\tau,x_0,y_0)-y_0(\tau,x_0,y_0)\|\right)d\tau\, ds$$

$$\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|\le\frac{(L_1+bL_2)\,t^{\alpha}}{\Gamma(\alpha+1)}\left(\|x_1(s,x_0,y_0)-x_0(s,x_0,y_0)\|+\|y_1(s,x_0,y_0)-y_0(s,x_0,y_0)\|\right)$$

and

$$\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\|\le\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g(s,x_1(s,x_0,y_0),y_1(s,x_0,y_0))-g(s,x_0(s,x_0,y_0),y_0(s,x_0,y_0))\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b \|k_2(s,\tau,x_1(\tau,x_0,y_0),y_1(\tau,x_0,y_0))-k_2(s,\tau,x_0(\tau,x_0,y_0),y_0(\tau,x_0,y_0))\|d\tau\, ds$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_3\left(\|x_1(s,x_0,y_0)-x_0(s,x_0,y_0)\|+\|y_1(s,x_0,y_0)-y_0(s,x_0,y_0)\|\right)ds+$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b L_4\left(\left\|x_1(\tau,x_0,y_0)-x_0(\tau,x_0,y_0)\right\|+\left\|y_1(\tau,x_0,y_0)-y_0(\tau,x_0,y_0)\right\|\right)d\tau\, ds$$

$$\left\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\right\|\le\frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)}\left(\left\|x_1(s,x_0,y_0)-x_0(s,x_0,y_0)\right\|+\left\|y_1(s,x_0,y_0)-y_0(s,x_0,y_0)\right\|\right)$$

So that

$$\left\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\right\|\le\frac{(L_1+bL_2)t^{\alpha}}{\Gamma(\alpha+1)}\left\|x_m(s,x_0,y_0)-x_{m-1}(s,x_0,y_0)\right\|+\frac{(L_1+bL_2)t^{\alpha}}{\Gamma(\alpha+1)}\left\|y_m(s,x_0,y_0)-y_{m-1}(s,x_0,y_0)\right\| \quad (7)$$

$$\left\|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\right\|\le\frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)}\left\|x_m(s,x_0,y_0)-x_{m-1}(s,x_0,y_0)\right\|+\frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)}\left\|y_m(s,x_0,y_0)-y_{m-1}(s,x_0,y_0)\right\| \quad (8)$$

Rewrite the inequalities (7) and (8) as a vectors:

$$\begin{pmatrix}\left\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\right\|\\ \left\|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\right\|\end{pmatrix}\le\begin{pmatrix}\frac{(L_1+bL_2)t^{\alpha}}{\Gamma(\alpha+1)} & \frac{(L_1+bL_2)t^{\alpha}}{\Gamma(\alpha+1)}\\ \frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)} & \frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)}\end{pmatrix}\begin{pmatrix}\left\|x_m(s,x_0,y_0)-x_{m-1}(s,x_0,y_0)\right\|\\ \left\|y_m(s,x_0,y_0)-y_{m-1}(s,x_0,y_0)\right\|\end{pmatrix}$$

or

$$V_{(m+1)\alpha}(t,x_0,y_0)\le\Lambda_\alpha(t)\,V_{m\alpha}(t,x_0,y_0) \quad (9)$$

$$V_{(m+1)\alpha}(t,x_0,y_0)=\begin{pmatrix}\left\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\right\|\\ \left\|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\right\|\end{pmatrix}$$

$$\Lambda_\alpha(t)=\begin{pmatrix}\frac{(L_1+bL_2)t^{\alpha}}{\Gamma(\alpha+1)} & \frac{(L_1+bL_2)t^{\alpha}}{\Gamma(\alpha+1)}\\ \frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)} & \frac{(L_3+bL_4)t^{\alpha}}{\Gamma(\alpha+1)}\end{pmatrix} \quad (10)$$

$$V_{m\alpha}\left(t,x_0,y_0\right)=\begin{pmatrix}\left\|x_m\left(s,x_0,y_0\right)-x_{m-1}\left(s,x_0,y_0\right)\right\| \\ \left\|y_m\left(s,x_0,y_0\right)-y_{m-1}\left(s,x_0,y_0\right)\right\|\end{pmatrix}$$

from (9) we get:

$$V_{(m+1)\alpha}\left(t\right)\leq\Lambda_{0\alpha}\left(t\right)V_{m\alpha}(t) \qquad (11)$$

Where $\Lambda_{0\alpha}=\max_{t\in\left[0,b\right]}\Lambda_{\alpha}(t)$. To prove the sequences (5) and (6) uniformly convergent , we must find the eigenvalue of the matrix (10) as the follows:

$$\begin{vmatrix}\frac{\left(L_1+bL_2\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}-\lambda_{\alpha} & \frac{\left(L_1+bL_2\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)} \\ \frac{\left(L_3+bL_4\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)} & \frac{\left(L_3+bL_4\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}-\lambda_{\alpha}\end{vmatrix}=0$$

$$\left(\frac{\left(L_1+bL_2\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}-\lambda_{\alpha}\right)\left(\frac{\left(L_3+bL_4\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}-\lambda_{\alpha}\right)-\frac{\left(L_1+bL_2\right)\left(L_3+bL_4\right)b^{2\alpha}}{\left(\Gamma\left(\alpha+1\right)\right)^2}=0$$

$$\lambda_{\alpha 1}=0 \quad \text{or} \quad \lambda_{\alpha 2}=\left(\frac{\left(L_3+bL_4\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\left(L_1+bL_2\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)$$

By iteration we find:

$$V_{(m+1)\alpha}\leq\Lambda_{0\alpha}^{m}V_{0\alpha} \quad , \quad V_{0\alpha}=\begin{pmatrix}\frac{\left(M_1+bM_2\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)} \\ \frac{\left(M_3+bM_4\right)b^{\alpha}}{\Gamma\left(\alpha+1\right)}\end{pmatrix} \qquad (12)$$

hence we obtain , $\sum_{i=1}^{m}V_{i\alpha}\leq\sum_{i=1}^{m}\Lambda_{0\alpha}^{i-1}V_{0\alpha}$ (13)

Since the matrix $\Lambda_{0\alpha}$ has greatest eigenvalue $\lambda_{\alpha\max}(\Lambda_{0\alpha})$ and by (4) we find the sequences (5) and (6) uniformly convergent to functions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ .i.e. :

$$\lim_{m\to\infty} x_m(t,x_0,y_0) = x(t,x_0,y_0)$$
$$\lim_{m\to\infty} y_m(t,x_0,y_0) = y(t,x_0,y_0)$$

Step 3: We have to show that $x(t,x_0,y_0) \in D_1^*$ and $y(t,x_0,y_0) \in D_2^*$, $\forall x_0 \in D_{1\alpha}, y_0 \in D_{2\alpha}, t \in [0,b]$

Indeed, it is enough to show that

$$\lim_{m\to\infty}\begin{pmatrix} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) + \int_0^b k_1(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0))\,d\tau\right]ds \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) + \int_0^b k_2(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0))\,d\tau\right]ds \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x(s,x_0,y_0),y(s,x_0,y_0)) + \int_0^b k_1(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\,d\tau\right]ds \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g(s,x(s,x_0,y_0),y(s,x_0,y_0)) + \int_0^b k_2(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\,d\tau\right]ds \end{pmatrix}$$

$$\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) - f(s,x(s,x_0,y_0),y(s,x_0,y_0))\right]ds + \right.$$

$$\left. + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b \left(k_1(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0)) - k_1(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\right)d\tau\, ds\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) - f(s,x(s,x_0,y_0),y(s,x_0,y_0))\right\|ds +$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^b \left\| k_1(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0)) - k_1(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)) \right\| d\tau\right) ds$$

$$\leq \frac{L_1+bL_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\| x_m(s,x_0,y_0) - x(s,x_0,y_0)\| + \| y_m(s,x_0,y_0) - y(s,x_0,y_0)\|\right) ds$$

$$\leq \frac{(L_1+bL_2)\, t^{\alpha}}{\Gamma(\alpha)}\left(\| x_m(s,x_0,y_0) - x(s,x_0,y_0)\| + \| y_m(s,x_0,y_0) - y(s,x_0,y_0)\|\right)$$

Similarly

$$\left\| \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) - g(s,x(s,x_0,y_0),y(s,x_0,y_0))\right] ds + \right.$$

$$\left. +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^b \left(k_2(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0)) - k_2(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\right) d\tau\, ds \right\|$$

$$\leq \frac{(L_3+bL_4)\, t^{\alpha}}{\Gamma(\alpha)}\left(\| x_m(s,x_0,y_0) - x(s,x_0,y_0)\| + \| y_m(s,x_0,y_0) - y(s,x_0,y_0)\|\right)$$

Since the sequences $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ uniformly convergent to the functions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ respectively on the interval $[0,b]$, then

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) + \int_0^b k_1(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0))\, d\tau\right] ds \xrightarrow{unif}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,x(s,x_0,y_0),y(s,x_0,y_0)) + \int_0^b k_1(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\, d\tau\right] ds$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) + \int_0^b k_2(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0))\, d\tau\right] ds \xrightarrow{unif}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g(s,x(s,x_0,y_0),y(s,x_0,y_0)) + \int_0^b k_2(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\, d\tau\right] ds$$

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Therefore

$$\lim_{m\to\infty}\begin{pmatrix}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f\left(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\right)+\int_0^b k_1\left(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0)\right)d\tau\right]ds \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g\left(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\right)+\int_0^b k_2\left(s,\tau,x_m(\tau,x_0,y_0),y_m(\tau,x_0,y_0)\right)d\tau\right]ds\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f\left(s,x(s,x_0,y_0),y(s,x_0,y_0)\right)+\int_0^b k_1\left(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)\right)d\tau\right]ds \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[g\left(s,x(s,x_0,y_0),y(s,x_0,y_0)\right)+\int_0^b k_2\left(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)\right)d\tau\right]ds\end{pmatrix}$$

We deduce that $\begin{pmatrix}x(t,x_0,y_0)\\ y(t,x_0,y_0)\end{pmatrix}$ is a solution for the system of nonlinear fractional integro-differential equations (1).

Theorem 3.2. (Uniqueness theorem)

Assume that the hypotheses A_1-A_4 are satisfying. Then the system of nonlinear fractional integro-differential equations (1) has a unique solution.

Proof : Let us consider $\begin{pmatrix}p(t,x_0,y_0)\\ q(t,x_0,y_0)\end{pmatrix}$ to be another solution of the system (1), that is

$$\begin{pmatrix}p(t,x_0,y_0)=c_1+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f\left(s,p(s,x_0,y_0),q(s,x_0,y_0)\right)+\int_0^b k_1\left(s,\tau,p(\tau,x_0,y_0),q(\tau,x_0,y_0)\right)d\tau\, ds\right] \\ q(t,x_0,y_0)=c_2+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f\left(s,p(s,x_0,y_0),q(s,x_0,y_0)\right)+\int_0^b k_2\left(s,\tau,p(\tau,x_0,y_0),q(\tau,x_0,y_0)\right)d\tau\, ds\right]\end{pmatrix}$$

hence we have

$$\left\| x(t,x_0,y_0) - p(t,x_0,y_0) \right\| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\left\| f(s,x(s,x_0,y_0),y(s,x_0,y_0)) - f(s,p(s,x_0,y_0),q(s,x_0,y_0)) \right\| +$$

$$+ \left\| \int_o^b k_1(s,\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0))\,d\tau - \int_0^b k_1(s,\tau,p(\tau,x_0,y_0),q(\tau,x_0,y_0))\,d\tau \right\| \Big] ds$$

$$\left\| x(t,x_0,y_0) - p(t,x_0,y_0) \right\| \le \frac{L_1 + bL_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| + \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \Big] ds$$

$$\Big[\left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| + \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \Big]$$

$$\left\| x(t,x_0,y_0) - p(t,x_0,y_0) \right\| \le \frac{(L_1 + bL_2)\,t^{\alpha}}{\Gamma(\alpha+1)} \Big[\left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| + \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \Big]$$

Similarly, we get:

$$\left\| y(t,x_0,y_0) - q(t,x_0,y_0) \right\| \le \frac{(L_3 + bL_4)\,t^{\alpha}}{\Gamma(\alpha+1)} \Big[\left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| + \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \Big]$$

So that

$$\begin{pmatrix} \left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| \\ \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \end{pmatrix} \le \Lambda_{0\alpha} \begin{pmatrix} \left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| \\ \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \end{pmatrix} \tag{15}$$

By iteration we have:

$$\begin{pmatrix} \left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| \\ \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \end{pmatrix} \le \Lambda_{0\alpha}^{m} \begin{pmatrix} \left\| x(s,x_0,y_0) - p(s,x_0,y_0) \right\| \\ \left\| y(s,x_0,y_0) - q(\tau,x_0,y_0) \right\| \end{pmatrix}$$

But From the condition (4), we have $\Lambda_{0\alpha}^{m} \to 0$ when $m \to \infty$, hence we obtain that:

$x(t,x_0,y_0) = p(t,x_0,y_0)$ and $y(t,x_0,y_0) = q(t,x_0,y_0)$. i.e. $\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$ is a unique solution for the system of nonlinear fractional integro-differential equations (1) on the domain (2).

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